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Branching point processes with independent transformations

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Abstract. We investigate branching point processes with an arbitrary number of point-types, based on the generating functional technique. Every point can be characterized by a set of parameters such as time, space, energy, etc. The analyses have been done for a random number of identical stages with independent transformations. The set of functional equations has been obtained for the determination of either the statistical characteristics of the resulting point process or the parameters of the separate stage process. The proposed approach lets us describe a wide range of random point processes and find mechanisms of their formation. We considered some specific models of electron multiplication in a microchannel plate, a photomultiplier tube and an avalanche photodiode.

1. Introduction

Processes of point multiplication take place in a variety of modern physics phenomena, such as photon and particle detectors, lasers, cosmic ray showers, neutron chain reactions, etc. Analysis of such processes has been performed by means of the mathematical technique of branching-point process theory [1]. The technique of investigation is based mainly on the application of generating functions. It lets us determine the statistical characteristics of the number of points in the region of multiplication. Thus, the amplitude distribution of the single electron signal of a photomultiplier has been calculated [2, 3]. Also, some particular methods have been suggested for determining the time characteristics of the branching point process. The evolution of cavity radiation with immigration and dead time was investigated in [4, 5]. Explicit expressions for the first two moments—intensity and correlation function—were provided. However, problems arise when we try to obtain time, space, energy, or any other characteristics of real physical objects participating in multiplication, simultaneously. For complete analysis of the branching point process it is worthwhile using generating functionals. The usefulness of such an approach was shown for the investigation of cascaded point processes [6, 7]. However, analysis has been undertaken only for the Poisson point process as a formalization of the process at separate stages of multiplication. Also, only time characteristics of the points were considered.

In this paper we do not restrict our investigation to any predefined type of random point process. We take into consideration only the common assumption of the independence of transformations of points at each stage of multiplication and consider that the laws of transformation are identical for all points participating in multiplication. Every point is determined by an arbitrary set of parameters. Under such conditions we managed to obtain the set of functional equations in a closed form. Each functional

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equation describes the distinct type of points and their transformations to other types. The procedure of functional differentiation gives the set of integral equations connecting particular statistical characteristics of a single stage and resulting processes, which can be solved analytically or numerically.

2. Generating functionals

A random point process is defined as a stochastic process with realizations, consisting of a collection of points, each one characterized by the vector coordinate ξ with an arbitrary set of components with well defined values. Let $\xi_1, \xi_2, \ldots$ be the occurrences of a point process in the region $\Omega = \Omega_1 \times \Omega_2 \times \ldots$, where Ω_i is the domain for the ith component. The probability generating functional (GFL) is defined as [8]

$$L[u;\Omega] = \left\langle \prod_{j=1}^{n} [1+u(\xi_j)] \right\rangle_{\xi,n}$$

where $u(t)$ $(-1 \leqslant u(t) \leqslant 0)$ is the trial function which is used as a formal parameter, $\langle \ldots \rangle_{\xi,n}$ represents averaging along the point parameters and the number of points in the Ω. After averaging, the last equation can be rewritten as

$$L[u;\Omega] = \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\Omega^i} \pi_i(\mu_1, \ldots, \mu_i, \Omega) \prod_{j=1}^{i} [1+u(\mu_j)]\, d\mu_1 \ldots d\mu_i$$

where $\pi_i(\mu_1, \ldots, \mu_i, \Omega)\, d\mu_1 \ldots d\mu_i$, $i = 0, 1, \ldots$, are the joint probabilities of obtaining squarely i points in the intervals $[\mu_1, \mu_1 + d\mu_1), \ldots, [\mu_i, \mu_i + d\mu_i) \in \Omega$. Also, symmetrical moment functions $f_i(\mu_1, \ldots, \mu_i)$, $i = 1, 2, \ldots$, are often used. The expressions $f_i(\mu_1, \ldots, \mu_i)\, d\mu_1 \ldots d\mu_i$, $i = 1, 2, \ldots$, are the joint probabilities of obtaining at least i points in the intervals $[\mu_1, \mu_1 + d\mu_1), \ldots, [\mu_i, \mu_i + d\mu_i) \in \Omega$. Furthermore, we shall call them correlation functions. The function of special interest is the intensity of random point process $f_1(\mu)$.

By means of functional differentiation [9] one can obtain different stochastic characteristics of the point process:

$$\pi_i(\mu_1, \ldots, \mu_i, \Omega) = \left. \frac{\delta^i L[u;\Omega]}{\delta u(t_1) \ldots \delta u(t_i)} \right|_{u(t)=-1} \tag{1}$$

$$f_i(\mu_1, \ldots, \mu_i) = \left. \frac{\delta^i L[u;\Omega]}{\delta u(t_1) \ldots \delta u(t_i)} \right|_{u(t)=0}. \tag{2}$$

The probability generation function (GF) is introduced as [10]

$$\theta[z,\Omega] = \langle z^n \rangle_n = \sum_{i=0}^{\infty} P_i(\Omega) z^i = L[z-1;\Omega] \tag{3}$$

where

$$P_i(\Omega) = \left. \frac{d^i \theta[z,\Omega]}{i!\, dz^i} \right|_{z=0} \tag{4}$$

is the probability of i points occurring in the Ω, and $z (0 \leqslant z \leqslant 1)$ is some auxiliary variable.

The models for the branching point process are based on the presentation of point processes as cluster random point processes [10]. Consider the formation of a cluster process. We have some initial point process (called A) in the region Ω. Each point of this primary process independently from the others gives birth to the individual secondary point process (called B_ξ, where ξ is the coordinate of the initiating point occurrence). All processes B_ξ are identical and differ from each other only by the coordinates of birth. Superposition of all secondary processes forms the cluster point process whose GFL can be written as [11]

$$L[u;\Omega]=L_A[L_B[u;\Omega|\cdot;\Omega]-1;\Omega] \tag{5}$$

where $L_A[v;\Omega]$ is the A process GFL, $L_B[u;\Omega|\tau;\Omega]$ is the B_τ process conditional GFL. Equation (5) can be extrapolated to any number of secondary point process generations when each point at each generation independently of the other points gives birth to the secondary point process.

3. Branching point process

In this section we consider a formalized model for the branching point process.

Every point is characterized by the vector coordinate of birth $\mu\in\Omega$ with an arbitrary set of components, by its type and by its capability to produce points at the next stage. Point type determines different laws of transformation and different sets of components in the point's vector coordinate participating in multiplication. It corresponds to the different physical nature of real objects in the multiplication process. The capability to produce points means that every point can generate a point process at the next stage (producing point) or not (unproducing point). We suppose that all points after some, generally speaking, random number of multiplication stages become unproducing with probability 1. Also, we are interested in the statistical characteristics of the point process without multiplication, that is, the process consisting of the unproducing points.

Let $L_A[v;\Omega;u;\Omega]$ be the GFL of the initiating point process, where $u=\{u_1(\mu),\dots,u_n(\mu)\}$ and $v=\{v_1(\nu),\dots,v_n(\nu)\}$ are the probe functions of producing and unproducing points, respectively, and n is the number of point types.

Consider the lth stage of multiplication. After $(l-1)$ stages, points with coordinates $\mu_{l-1}^{1,i}$, $\mu_{l-1}^{2,i}$, ... (producing points), $\nu_{l-1}^{1,i}$, $\nu_{l-1}^{2,i}$, ... (unproducing points), $i=1,\dots,n$, have been generated. Their GFL is $L_{l-1}[v;\Omega;u;\Omega]$. Every point with coordinate $\mu_{l-1}^{j,i}$ produces a random number of producing points of all types, whose coordinates are $\mu_l^{k,p|j,i}$, $p=1,\dots,n$; $k=1,\dots$, or transform into an unproducing point with coordinate $\nu_l^{m,q|j,i}$, $q=1,\dots,n$; $m=1,\dots$, independently of the other points born at the $(l-1)$th stage. Their GFL is $L_0^j[v;\Omega;u;\Omega|\mu_{l-1}^{j,i}]$. So the process of transformation consists of either generation of the producing points or transformation into the unproducing point. The laws of generation and transformation for each point of the $(l-1)$th stage are identical and differ from each other only by the point's coordinate.

The resulting process after l stages consists of producing points of all types, whose coordinates are $\mu_l^{j,i}$, $j=1,2,\dots$, $i=1,\dots,n$, and unproducing points of all types after $(l-1)$ stages of multiplication, whose coordinates are $\nu_r^{j,i}$, $j=1,2,\dots$, $i=1,\dots,n$, $r=1,\dots,l$. According to equation (5), the GFL of the resulting process after l stages is given by

$$L_l[v;\Omega;u;\Omega]=L_{l-1}[v;\Omega;L_0^1[v;\Omega;u;\Omega|\mu_{l-1}^1]-1;\Omega_1;\dots;L_0^n[v;\Omega;u;\Omega|\mu_{l-1}^n]-1;\Omega_n]$$

where GFL $L_{l-1}[v;\Omega;u;\Omega]$ can be written as

$$L_{l-1}[v;\Omega;u;\Omega]=L_{l-1}[v;\Omega;u_1(\mu);\Omega_1;u_2(\mu);\Omega_2;\ldots;u_n(\mu);\Omega_n]$$

and high index j in μ_l^{ji}, ν_l^{ji} is omitted.

As the laws of generation and transformation are independent of the number of stage, that is, the points at each stage are the result of identical multiplications, the last equation takes the form

$$\begin{aligned}
&L_l[v;\Omega;u;\Omega]\\
&\quad=L_1[v;\Omega;\\
&\quad L_0^1[v;\Omega;\ldots\\
&\quad L_0^1[v;\Omega;L_0^1[v;\Omega;u;\Omega|\mu_{l-1}^1]-1;\Omega_1];\ldots;\\
&\quad L_0^n[v;\Omega;u;\Omega|\mu_{l-1}^n]-1;\Omega_n|\mu_{l-2}^1]-1;\Omega_1];\\
&\quad\vdots\\
&\quad L_0^n[v;\Omega;L_0^1[v;\Omega;u;\Omega|\mu_{l-1}^1]-1;\Omega_1];\ldots;\\
&\quad L_0^n[v;\Omega;u;\Omega|\mu_{l-1}^n]-1;\Omega_n|\mu_{l-2}^n]-1;\Omega_n|\ldots\mu_1^1]-1;\Omega_1];\\
&\quad\vdots\\
&\quad L_0^n[v;\Omega;\ldots\\
&\quad L_0^1[v;\Omega;L_0^1[v;\Omega;u;\Omega|\mu_{l-1}^1]-1;\Omega_1];\ldots;\\
&\quad L_0^n[v;\Omega;u,\Omega|\mu_{l-1}^n]-1;\Omega_n|\mu_{l-2}^1]-1;\Omega_1];\\
&\quad\vdots\\
&\quad L_0^n[v;\Omega;L_0^1[v;\Omega;u;\Omega|\mu_{l-1}^1]-1;\Omega_1];\ldots;\\
&\quad L_0^n[v;\Omega;u;\Omega|\mu_{l-1}^n]-1;\Omega_n|\mu_{l-2}^n]-1;\Omega_n|\ldots\mu_1^n]-1;\Omega_n]
\end{aligned}\tag{6}$$

where $L_1[v;\Omega;u;\Omega]$ is the GFL of initiating points.

Denoting the GFL of points after l stages, initiated by the point of type i with coordinate μ^i, by $Q_l^i[v;\Omega;u;\Omega|\mu^i]$, we have

$$\begin{aligned}
&Q_l^i[v;\Omega;u;\Omega|\mu^i]\\
&\quad=L_0^i[v;\Omega;Q_{l-1}^1[v;\Omega;u;\Omega|\cdot]-1;\Omega_1;\ldots;Q_{l-1}^n[v;\Omega;u;\Omega|\cdot]-1;\Omega_n|\mu^i]\\
&\quad i=1,\ldots,n.
\end{aligned}\tag{7}$$

If, after some multiplication stages, all points become unproducing, one can obtain

$$\lim_{l\to\infty}L_l[v;\Omega;u;\Omega]=L_r[v;\Omega]\tag{8}$$

where $L_r[v;\Omega]$ is the GFL of the resulting point process. The same expression takes place for the GFL $Q_l^i[v;\Omega;u;\Omega|\mu^i]$, $i=1,\ldots,n$:

$$\lim_{l\to\infty}Q_l^i[v;\Omega;u;\Omega|\mu^i]=Q^i[v;\Omega|\mu^i].\tag{9}$$

In this case, $Q^i[v;\Omega|\mu]$ can be interpreted as the GFL of the resulting point process, initiated by the point of type i with birth coordinate μ^i.

Letting l go to infinity in equation (7) and taking into account equation (9), we have

$$Q^i[v;\Omega|\mu^i]=L_0^i[v;\Omega;Q^1[v;\Omega|\cdot]-1;\Omega_1;\ldots;Q^n[v;\Omega|\cdot]-1;\Omega_n|\mu^i]$$
$$i=1,\ldots,n. \tag{10}$$

Combining equations (6) and (8) yields

$$L_r[v;\Omega]=L_1[v;\Omega;Q^1[v;\Omega|\cdot]-1;\Omega_1;\ldots;Q^n[v;\Omega|\cdot]-1;\Omega_n]. \tag{11}$$

Equations (10) and (11) are the set of functional equations for obtaining different statistical characteristics of the resulting point process when parameters of the separate stage process are known and vice versa—i.e. to get the statistical parameters of the separate stage process based on the characteristics of the resulting process.

There are a few situations when we can evaluate equation (10) directly. Mostly we have to make use of functional differentiation (equations (1) or (2)) of equation (10) up to the necessary degree when $v=0$ or $v=-1$. This technique gives integral equations, containing correspondent stochastic characteristics of single stage points of any type and resulting process.

Equations for the GF (equation (3)) can be obtained from equations (10) and (11) by substituting variable $z-1$ for function v:

$$\theta_i[z;\Omega|\mu^i]=L_0^i[z-1;\Omega;\theta_1[z;\Omega|\cdot]-1;\Omega_1;\ldots;\theta_n[z;\Omega;|\cdot]-1;\Omega_n|\mu^i]$$
$$i=1,\ldots,n. \tag{12}$$

$$\theta_r[z;\Omega]=L_1[z-1;\Omega;\theta_1[z;\Omega|\cdot]-1;\Omega_1;\ldots;\theta_n[z;\Omega|\cdot]-1;\Omega_n]. \tag{13}$$

$\theta_i[z;S|\mu]$ can be determined by numeric evaluation of equation (12) when GFL $L_0^i[v;\Omega;u;\Omega|\mu^i]$, $i=1,\ldots,n$, is known.

Furthermore, we shall restrict our consideration to the case when points of only one type can take part in multiplication. So $n=1$ and the set of equations for GFL (10), (11) takes the form

$$Q[v;\Omega|\mu]=L_0[v;\Omega;Q[v;\Omega|\cdot]-1;\Omega|\mu] \tag{14}$$

$$L_r[v;\Omega]=L_1[v;\Omega;Q[v;\Omega|\cdot]-1;\Omega]. \tag{15}$$

For GF (12), (13) one can write

$$\theta[z;\Omega|\mu]=L_0[z-1;\Omega;\theta[z;\Omega|\cdot]-1;\Omega|\mu] \tag{16}$$

$$\theta_r[z;\Omega]=L_1[z-1;\Omega;\theta[z;\Omega|\cdot]-1;\Omega]. \tag{17}$$

It is clear that all results obtained in this case can be extrapolated to the situation with any number of point types.

4. Devices with secondary electron multiplication

Microchannel plates (MCP), photomultiplier tubes (PMT), and avalanche photodiodes (APD) are widely used in a variety of systems for the transformation of an optical field into photoelectron flow and further amplification of the signal. Processes of amplification, based on the avalanche multiplication of electrons, naturally lead to the distortion of the output signal. Up to now, particular analytical results have been obtained only for some special cases of electron multiplication in the channel of MCP [12–15], PMT

[16-18] and APD [19-21]. The main problem of the theoretical investigation of functional properties of such devices generally involves the continuous amplification region, and the random number of multiplication cascades (i.e. the output signal is formed by the electrons released at each cascade). The usual technique, based on the generating functions, sometimes gives reasonable results for the conventional PMT's where the multiplication system is discrete with a definite number of dynodes; however, for the analysis of MCP or APD multiplication it is not quite as well suited.

The detection process begins when the initiating particle (i.e. photoelectron, photon, ion, etc) is directed to the amplification region of the device. It can be the channel surface of the MCP, dynode system of the PMT or the depletion layer of the junction of the APD. There the initiating particle produces some secondary electrons. Each one, accelerated by the applied field, strikes the channel wall in the MCP or the next dynode in the PMT or ionizes atoms in the APD, producing some secondaries itself. This process is repeated many times while passing through the amplification region until finally the electron avalanche reaches the anode or output electrode.

Functional equations (14) and (15) let us determine the output signal characteristics if the characteristics of electrons at a single cascade and the characteristics of the initiating particles are known. We can make use of functional differentiation of equations (14) and (15) with respect to v up to the necessary degree when $v=0$ or $v=1$. This technique gives integral equations, containing corresponding stochastic characteristics of the output electrons, electrons ejected after one cascade, and initiating particles.

Furthermore, we shall accept that electrons in the amplification region can be introduced as producing points and electrons that have left it as unproducing points. In this case, producing and unproducing points are distinguished by their space coordinates. So we can connect the probe function of producing points u with the amplification region, termed G, and the probe function v of unproducing points with the output electrode or anode region, termed S.

As one can see, equations (14) and (15) can be solved if the single stage GFL $L_0[v; S; u; G|\mu]$ is known. It can be written for the considered earlier multiplication processes as

$$L[v;S;u;G|\mu]=\int_G h(\nu|\mu)\langle[1+u(\nu)]^{m(\nu-\mu)}\rangle_{m(\nu-\mu)}\,d\nu+\int_S h(\nu|\mu)[1+v(\nu)]\,d\nu \qquad (18)$$

where $h(\nu|\mu)$ is the probability density that the electron, born at the point with coordinate μ, hits the point with coordinate ν; $\langle\ldots\rangle$ is the averaging along the number of electrons $m(\nu-\mu)$, whose birth coordinate is ν, ejected by the electron with birth coordinate μ. The first term in the right-hand side of equation (18) describes the collision of electrons (point process of multiple points) and the second term describes electrons leaving the region of multiplication.

The GF of the number of output electrons $\theta[z; S|\mu]$ can be determined by numerical evaluation of (16). Taking into account equation (18), equation (16) can be rewritten as

$$\theta[z;S|\mu]=\int_G h(\nu|\mu)\langle\theta^{m(\nu-\mu)}[z;S|\nu]\rangle_{m(\nu-\mu)}\,d\nu+z\int_S h(\nu|\mu)\,d\nu. \qquad (19)$$

It is a nonlinear integral equation, which has no analytical solution in the common case. Amplitude characteristics can be found by differentiation of equation (19) with respect to the variable z when $z=0$ or $z=1$.

5. Microchannel plate

As the results for the gain and time distribution of the single electron signal for the MCP have been obtained previously [15], in this section we are going to discuss problems of pulse-height distribution calculation.

In this case, coordinate μ can be still considered as a set of components such as space, time, energy, and radial or azimuthal coordinates. The producing or unproducing property of the electron is determined only by its space coordinate along the channel axis.

The pulse-height distribution of the output electron signal can be evaluated from equation (16). By means of differentiation with respect to z when $z=0$, one can obtain the probabilities $P(n|\mu)$ of the emerging n electrons at the channel output, initiated by the electron with birth coordinate μ:

$$P(0|\mu)=\int_G h(\nu|\mu)F[\nu-\mu, P(0|\nu)]\,\mathrm{d}\nu$$

$$P(1|\mu)=\int_G W(\nu,\mu)P(1|\nu)\,\mathrm{d}\nu+\int_S h(\nu|\mu)\,\mathrm{d}\nu \tag{20}$$

$$P(n|\mu)=\int_G W(\nu,\mu)P(n|\nu)\,\mathrm{d}\nu+S(n|\mu) \qquad n\geqslant 2$$

where

$$S(n|\mu)=\int_G h(\nu|\mu)\,\mathrm{d}\nu \sum_{j_1 i_1+\ldots+j_p i_p=n, i_p\neq n} \prod_{r=1}^{p}\frac{1}{j_r!}P^{j_r}(i_r|\nu)F^{x}_{p(0|\nu)}[\nu-\mu, P(0|\nu)]$$

$$W(\nu,\mu)=h(\nu|\mu)F'_{p(0|\nu)}[\nu-\mu, P(0|\nu)]$$

$$F^{x}_{p(0|\nu)}[\nu-\mu, P(0|\nu)]=\frac{\mathrm{d}^x F[\nu-\mu, P(0|\nu)]}{\mathrm{d}[P(0|\nu)]^x}$$

for $n\geqslant 2$, where $x=\Sigma_{r=1}^{p} j_r$, sum $j_1 i_1+\ldots+j_p i_p=n$ means picking out all combinations of values of the indexes j_r, i_r, $r=1,\ldots,p$, producing in sum the value n, where $i_k\neq i_m$, when $k\neq m$, $k, m=1,\ldots,p$.

The recurrent expressions (20) let us obtain the pulse-height probabilities for any kind of single cascade distributions. In order to obtain the resulting expressions for the pulse-height distribution, one has to average these equations along the initiating particle coordinates and the number of electrons ejected from the channel wall by the initiating particle. It can be done also by means of differentiating equation (17) with respect to z when $z=0$, in the same way as equations (20) were obtained.

If the single cascade pulse-height distribution is given by the Poisson law [10], so

$$\langle[1+u(\nu)]^{m(\nu-\mu)}\rangle_{m(\nu-\mu)}=\exp\{\kappa(\nu-\mu)u(\nu)\}$$

$$F[\nu-\mu, u(\nu)]=\exp\{\kappa(\nu-\mu)u(\nu)\}$$

$$F^{x}_{u(\nu)}[\nu-\mu, u(\nu)]=\{\kappa(\nu-\mu)\}^x\exp\{\kappa(\nu-\mu)u(\nu)\}$$

where $\kappa(\nu-\mu)$ is the average number of the electrons $m(\nu-\mu)$ whose birth coordinate is ν, ejected by the electron with birth coordinate μ. Probabilities $P(n|\mu)$ of the emerging n electrons at the channel output, initiated by the electron with birth coordinate μ, yield

$$P(n|\mu)=\int_G h(\nu|\mu)H(n|\nu,\mu)\,\mathrm{d}\nu \qquad n\geqslant 2$$

where

$$H(n|\nu,\mu)=\frac{\kappa(\nu-\mu)}{n}\sum_{i=1}^{n-1} iH(n-i|\nu,\mu)\int_G h(\xi|\mu)H(i|\xi,\nu)\,\mathrm{d}\xi$$

$$+\kappa(\nu-\mu)H(0|\nu,\mu)\int_G h(\xi|\mu)H(n|\xi,\nu)\,\mathrm{d}\xi. \tag{21}$$

The solution of the integral equation (21) can be obtained, for example, by means of the numerical iteration method.

6. Photomultiplier tube

A lot of information devoted to the analysis of PMT can be found in the literature. However, only the simplest cases of electron multiplication allow us to obtain analytical models in the closed form, convenient for further calculations. Effects such as the losses of electrons in the dynode system, transit of an electron by the dynode, and non-Poisson single dynode statistics, are still difficult to investigate. Here we try to apply the previously introduced technique for the description of the conventional PMT. We shall assume that each electron possesses two components of coordinate: time and the number of the dynode where it was born.

Conventional PMT single-stage GFL can be written as

$$L_0[v;S;u;G|k,\gamma]$$

$$=\sum_{j=1}^{N} p(j|k)\int_U w(\varphi|\gamma,k,j)\times\langle(1+u(j,\varphi))^{m(k,j,\varphi,\gamma)}\rangle_{m(k,j,\varphi,\gamma)}\,\mathrm{d}\varphi$$

$$+p(N+1|k)\int_U w(\varphi|\gamma,k,N+1)(1+v(k,\varphi))\,\mathrm{d}\varphi \tag{22}$$

where N is the number of dynodes, $L_0[v;S;u;G|k,\gamma]$ is the GFL of the output electrons, produced by the electron born at the kth dynode at time γ, $p(j|k)$ is the probability of the electron, born at the kth dynode, striking the dynode of number j (the $N+1$ dynode is the anode), $w(\varphi,|\gamma,k,j)$ is the time probability distribution of obtaining an electron at instant φ at the dynode j, provided it was at instant γ at the dynode k. This GFL describes the electron flow from one dynode to the others and takes into account the electron losses in the dynode system and transit by dynodes.

As an example, we now consider the problem of determining the PMT single-electron signal time distribution. By differentiating equation (14) with respect to v, taking into consideration equation (22), we can obtain the set of integral equations for the intensity

function of electrons at the output, produced by a single electron from the mth dynode, born at instant γ:

$$g(t|\gamma, m) = \sum_{j=1}^{N} f(j|m) \int_0^t w(\varphi|\gamma, m, j) g(t|\varphi, j)\, d\varphi$$
$$+ p(N+1|m) w(t|\gamma, m, N+1) \qquad m = 1, 2, \ldots \tag{23}$$

where $f(j|m)$ is the intensity of the electrons at the jth dynode, ejected by one electron from the mth dynode. Denoting $w(\varphi|\gamma, m, j) = w(\varphi - \gamma|m, j)$, $g(t|\varphi, j) = g(t-\varphi|j)$, Laplace transformation of (23) gives

$$g(s|m) = \sum_{j=1}^{N} f(j|m) w(s|m, j) g(s|j)$$
$$+ p(N+1|m) w(s|m, N+1) \qquad m = 1, \ldots, N.$$

This set of linear algebraic equations can be solved and the Laplace transformation of the time intensity of the output signal from every dynode $m = 1, \ldots, N$ obtained. Let us consider some examples with an analytical solution of this system. Assuming that the transit of dynodes is absent and characteristics of multiplication are independent of the number of dynode, i.e.

$$f(m+i|m) = 0 \qquad i = 0, 2, 3, \ldots, m = 1, \ldots, N$$
$$f(m+1|m) = f \qquad w(s|m, m+1) = w(s) \qquad m = 1, \ldots, N$$

the last equation takes the form

$$g(s|m) = f w(s) g(s|m+1) \qquad m = 1, \ldots, N.$$

Its solution is

$$g(s|m) = [f w(s)]^{N-m+1}.$$

For the second multiplier in the right-hand side of the last expression, the inverse Laplace transform gives $N-m+1$-fold convolution of probability distribution function w. The exact analytical expression for $g(\tau|m)$ can be obtained when $w(\tau)$ is known. Let us put

$$w(\tau) = r/(b-a)\{\exp(-a\tau) - \exp(-b\tau)\}$$

and $w(s) = r/(s+a)/(s+b)$, where r, a, b are the characteristics of the single-dynode electron time distribution. Then the resulting time distribution will be written as

$$g(t|1) = (pr)^{N-1} \sum_{m=1}^{N} \left\{ \prod_{k=0}^{m-1} (-N+k) \frac{(b-a)^{-N+1-m} t^{N-m}}{(N-m)!(m-1)!} \right.$$
$$\left. \times [\exp(-at) + (-1)^{-N+1-m} \exp(-bt)] \right\}.$$

7. Avalanche Photodiode

The statistical nature of the multiplication processes in the depletion layer of an APD gives rise to the high level of APD noise. Noise characteristics of the APD in the main

determine functional properties of the device and should be taken into account when electronic equipment of detectors is designed. The noises of APD, as a rule, are characterized by noise factor F, introduced as

$$F = M_2/M_1^2$$

where M_1 is the APD mean value of gain and M_2 is its second-order moment.

The general model of branching point processes, presented in section 3, let us build an analytical model of APD electron multiplication without any restriction to the type of probability distribution for ionized particles in the amplification region.

For the determination of the noise factor we can retain in our model only the linear space coordinate of the electron birth. We define $q(y|x)$ as the probability density function of the distance (flight of the electron) between two points of ionization y and x. Single-stage multiplication GFL in this case can be written as

$$L_0[v; S; u; G|x] = \int_G q(y|x)[1+u(y)]^2\,dy + \int_S q(y|x)[1+v(y)]\,dy. \tag{24}$$

Combining equations (16) and (24) yields

$$\theta[z|x] = \int_x^L g(y|x)\theta^2[z|y]\,dy + z\int_L^\infty g(y|x)\,dy. \tag{25}$$

Equation (25) is an integral equation for the determination of GF of the electrons, obtained at the output electrode, initiated by an electron with birth coordinate x. Differentiating equation (25) with respect to z, we can obtain the moments of any order for output electrons. For example, the first-order moment takes the form

$$M_1(x) = \theta'(z|x)|_{z=1} = 2\int_x^L g(y-x)M_1(y)\,dy + \int_L^\infty g(y-x)\,dy \tag{26}$$

and the second-order moment

$$\begin{aligned} M_2(x) &= \theta''(z|x)|_{z=1} + M_1(x) \\ &= 2\int_x^L g(y-x)M_2(y)\,dy + 2\int_x^L g(y-x)[M_1(y)]^2\,dy + \int_L^\infty g(y-x)\,dy. \end{aligned} \tag{27}$$

Integral equations (26) and (27) are fundamental to the determination of amplification characteristics of the signal in an APD. In order to obtain the resultant expressions for M_1 and M_2 we have to average these equations along the initiating particle space coordinate and specify kernel function $g(y-x)$.

If the distance between two ionizations is exponentially distributed, i.e.

$$g(y) = a\exp\{-ay\} \qquad a, y > 0$$

then

$$M_1(x) = \exp\{a(L-x)\}$$

$$M_2(x) = \exp\{a(L-x)\}[2\exp\{a(L-x)\} - 1]$$

and the noise factor takes the form

$$F(x) = M_2(x)/M_1^2(x) = 2 - \exp\{-ax\}.$$

8. Conclusion

In this paper we have built up the mathematical model for the branching point process with independent transformations for a random number of identical stages which lets us obtain either the statistical characteristics of the resulting point process or the parameters of the separate stage process. It is supposed that the scheme of branching is known. This model can be applied to a wide range of phenomena in nuclear physics, optics, biology, etc.

The approach offered was applied to the analysis of secondary electron amplification in devices with secondary electron multiplication: microchannel plates, photomultipliers and avalanche photodiodes.

The technique described is also perfectly suited for the analysis of a variety of processes in quantum optics and the designing of computer simulators of random point processes. These problems are currently also under our consideration.

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